Test of Mathematics for University Admission (TMUA)
2023 Paper 1 Worked Solutions

Version 1.0, September 2023

Contents

Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the TMUA Preparation Materials

Question 1

We calculate the integrals in terms of a and b:

$$\begin{aligned}
\int_0^1 (ax+b)\,\mathrm{d}x &= \left[\tfrac{1}{2}ax^2 + bx\right]_0^1 \\
&= \tfrac{1}{2}a + b \\
&= 1 \\
\int_0^1 x(ax+b)\,\mathrm{d}x &= \int_0^1 (ax^2 + bx)\,\mathrm{d}x \\
&= \left[\tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2\right]_0^1 \\
&= \tfrac{1}{3}a + \tfrac{1}{2}b \\
&= 1
\end{aligned}$$

Multiplying the first equation by 2 and the second by 6 gives the simultaneous equations

$$\begin{aligned}
a + 2b &= 2 \qquad (1) \\
2a + 3b &= 6 \qquad (2)
\end{aligned}$$

Then $2 \times (1) - (2)$ gives $b = -2$, so $a = 6$. Therefore $a + b = 4$ and the correct answer is option F.

Question 2

As the graphs do not meet, the simultaneous equations $y = x^2 + 5x + 6$ and $y = mx - 3$ have no solutions. Therefore $x^2 + 5x + 6 = mx - 3$ has no solutions, so the quadratic

$$x^2 + (5 - m)x + 9 = 0$$

has no solutions and must therefore have a negative discriminant. This gives

$$(5 - m)^2 - 4 \times 1 \times 9 = 25 - 10m + m^2 - 36 = m^2 - 10m - 11 < 0.$$

Factorising this quadratic in m gives

$$(m - 11)(m + 1) < 0.$$

This quadratic has roots -1 and 11, and as the coefficient of m^2 in $m^2 - 10m - 11$ is positive, the quadratic in m is negative between the roots. So the complete range of possible values of m is $-1 < m < 11$, which is option A.

Question 3

We work out the values of each of the individual integrals by splitting them into unit intervals:

$$\begin{aligned}
\int_0^3 f(x)\,\mathrm{d}x &= \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\
&= 1 + 2 + 3 \\
&= 6 \\
\int_1^3 f(x)\,\mathrm{d}x &= \int_1^2 f(x)\,\mathrm{d}x + \int_2^3 f(x)\,\mathrm{d}x \\
&= 2 + 3 \\
&= 5 \\
\int_2^3 f(x)\,\mathrm{d}x &= 3 \\
\int_4^3 f(x)\,\mathrm{d}x &= -\int_3^4 f(x)\,\mathrm{d}x \\
&= -4 \\
\int_5^3 f(x)\,\mathrm{d}x &= -\int_3^5 f(x)\,\mathrm{d}x \\
&= -\left(\int_3^4 f(x)\,\mathrm{d}x + \int_4^5 f(x)\,\mathrm{d}x\right) \\
&= -(4 + 5) \\
&= -9
\end{aligned}$$

where for the final two integrals, we have used the rule $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$.

Adding these gives the required sum:

$$6 + 5 + 3 - 4 - 9 = 1$$

which is option C.

Question 4

To help us understand what is going on here, we will work out the first few terms.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\sin\left(n\pi + \frac{\pi}{3}\right)}{2^n} &= \frac{\sin(\frac{\pi}{3})}{2^0} + \frac{\sin(\pi + \frac{\pi}{3})}{2^1} + \frac{\sin(2\pi + \frac{\pi}{3})}{2^2} + \frac{\sin(3\pi + \frac{\pi}{3})}{2^3} + \cdots \\ &= \frac{\sqrt{3}/2}{1} + \frac{-\sqrt{3}/2}{2} + \frac{\sqrt{3}/2}{4} + \frac{-\sqrt{3}/2}{8} + \cdots \\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\right)\end{aligned}$$

We see that the part in the brackets is a geometric series with first term $a = 1$ and common ratio $r = -\frac{1}{2}$, so its sum is

$$\frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{3/2} = \frac{2}{3}$$

and hence the sum of the original series is

$$\frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{\sqrt{3}}{3}$$

which is option C.

Question 5

There are (at least) two ways to calculate the area of the rectangle $RSTU$: we could find the lengths of the sides RU and RS and then multiply them, or we could find the area of the square $MNOP$ and subtract the areas of the shaded triangles. We demonstrate both approaches.

We ignore the units, taking lengths to be in cm and areas to be in cm^2.

We first note that the symmetry of the shape means that $MU = MR$, $NR = NS$ and so on.

Also, as the perimeter of $MNOP$ is 40, the side length is 10, so $NR = 10 - x$.

Approach 1: Finding the lengths of the rectangle's sides

The triangle RMU is right-angled, with $MR = MU = x$. By Pythagoras's Theorem, this means that $RU^2 = x^2 + x^2 = 2x^2$, so $RU = x\sqrt{2}$.

Now $NR = 10 - x$, so by a similar argument, $RS = (10 - x)\sqrt{2}$.

Multiplying these gives the area as $A = x\sqrt{2} \times (10 - x)\sqrt{2} = 2x(10 - x)$.

Approach 2: Finding the area by subtraction

The area of $MNOP$ is $10^2 = 100$. The area of MRU and the area of OST are each $\frac{1}{2}x^2$, so the total area of these two triangles is x^2. Likewise, the area of NRS and the area of PTU are each $\frac{1}{2}(10 - x)^2$, so the total area of these two triangles is $(10 - x)^2$. Therefore the area of the rectangle $RSTU$ is

$$100 - x^2 - (10 - x)^2 = 100 - x^2 - (100 - 20x + x^2) = 20x - 2x^2 = 2x(10 - x).$$

We are told that $A = 20$, so we have to solve the equation $2x(10 - x) = 20$. Expanding and rearranging gives $2x^2 - 20x + 20 = 0$; dividing by 2 simplifies this to $x^2 - 10x + 10 = 0$. We can use the quadratic formula to find the possible values of x; they are

$$x = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 10}}{2} = \frac{10 \pm \sqrt{60}}{2} = 5 \pm \sqrt{15}.$$

The larger of these is $5 + \sqrt{15}$, and so the answer is option F.

Question 6

A coefficient is divisible by 12 if and only if it is a multiple of $2^2 = 4$ and a multiple of 3. The coefficient of x^k (for $0 \leq k \leq 12$) is given by $\binom{12}{k} 2^{12-k} 3^k$.

This is certainly a multiple of 3 when $k \geq 1$.

This is certainly a multiple of $2^2 = 4$ when $k \leq 10$.

We therefore only need to check $k = 0$, $k = 11$ and $k = 12$.

When $k = 0$, the coefficient is $\binom{12}{0} \times 2^{12} = 2^{12}$, which is not a multiple of 3, and so not of 12 either.

When $k = 11$, the coefficient is $\binom{12}{11} \times 2 \times 3^{11}$. But $\binom{12}{11} = \binom{12}{1} = 12$, so this is a multiple of 12.

When $k = 12$, the coefficient is $\binom{12}{12} \times 3^{12} = 3^{12}$, which is not a multiple of 2, and so not of 12 either.

Therefore the coefficient is a multiple of 12 for $1 \leq k \leq 11$, so 11 terms have such a coefficient, and the answer is option E.

Question 7

Note that P(x) and Q(x) being in the ratio $4:1$ means that $\mathrm{P}(x) = 4\mathrm{Q}(x)$, so

$$2^x + 4 = 4\big(2^{(2x-2)} - 2^{(x+2)} + 16\big).$$

Simplifying the powers, noting that $4 = 2^2$, gives

$$2^x + 4 = 2^{2x} - 2^{x+4} + 64.$$

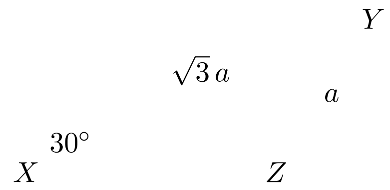
This is a quadratic in 2^x, so let us write $u = 2^x$. The equation can then be written as

$$u + 4 = u^2 - 16u + 64.$$

This rearranges to $u^2 - 17u + 60 = 0$, which factorises as $(u-5)(u-12) = 0$, so $u = 5$ or $u = 12$. Therefore, as $u = 2^x$, we have $x = \log_2 5$ or $x = \log_2 12$, and the largest possible value of x is $\log_2 12$, which is option F.

Question 8

Let us sketch a fun triangle.



The area of XYZ is given by the '$\frac{1}{2}ab\sin C$' formula: it is $\frac{1}{2}XY.XZ\sin 30^\circ = \frac{1}{4}\sqrt{3}\,a.XZ$. We must therefore find the possible values of XZ.

We can use the cosine rule to find XZ: we have

$$YZ^2 = XY^2 + XZ^2 - 2XY.XZ\cos 30^\circ.$$

Filling in the known values, we obtain

$$a^2 = 3a^2 + XZ^2 - 3a.XZ$$

so XZ satisfies the quadratic equation

$$XZ^2 - 3a.XZ + 2a^2 = 0.$$

This factorises to give

$$(XZ - 2a)(XZ - a) = 0$$

so either $XZ = a$ or $XZ = 2a$.

It is useful to think about what the two resulting triangles look like and whether these values of XZ make sense.

We are asked to find the ratio of the areas of these possible fun triangles. Since the area is given by $\frac{1}{4}\sqrt{3}\,a.XZ$, the ratio of areas is given by the ratio of possible XZ values. Since triangle S has the larger area, it must have the larger of the XZ values, so the ratio is $2a : a$, which equals $2 : 1$. The correct answer is thus option B.

> ***Commentary:*** *There are lots of appearances of XY, YZ and XZ here, following the language used in the question. It might have been simpler to use the standard side-length naming convention, letting x be the length of the side YZ opposite X, letting y be the length of the side XZ opposite Y and z the length of XY. Then we would have the area of XYZ being $\frac{1}{2}yz\sin 30^\circ = \frac{1}{4}\sqrt{3}\,ay$, the cosine rule would read*
>
> $$x^2 = y^2 + z^2 - 2yz\cos 30^\circ,$$
>
> *and we would finally deduce that $y = a$ or $y = 2a$.*

Question 9

We have $(1 + 3\cos 3\theta)^2 = 4$ if and only if $1 + 3\cos 3\theta = \pm 2$. We consider each possibility separately.

We have

$$\begin{aligned} & 1 + 3\cos 3\theta = 2 \\ \text{if and only if} \quad & 3\cos 3\theta = 1 \\ \text{if and only if} \quad & \cos 3\theta = \tfrac{1}{3}. \end{aligned}$$

Since $0^\circ \leq \theta \leq 180^\circ$, we have $0^\circ \leq 3\theta \leq 540^\circ$, and there are 3 values of 3θ which have $\cos 3\theta = \frac{1}{3}$ in this interval. (One is between 0° and 90°, one is between 270° and 360°, and one is between 360° and 450°, by considering the graph of $y = \cos x$.)

Now considering the other possibility, we have

$$\begin{aligned} & 1 + 3\cos 3\theta = -2 \\ \text{if and only if} \quad & 3\cos 3\theta = -3 \\ \text{if and only if} \quad & \cos 3\theta = -1. \end{aligned}$$

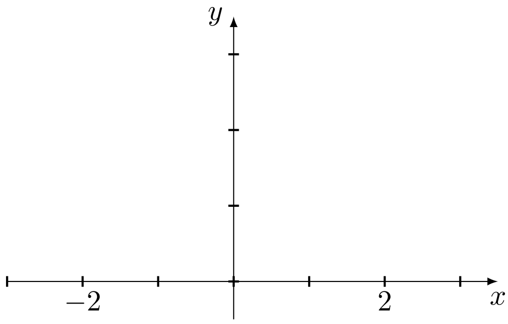
Again, $0^\circ \leq 3\theta \leq 540^\circ$, but this time there are only 2 values of 3θ which satisfy the equation: $3\theta = 180^\circ$ and $3\theta = 540^\circ$.

Neither of these values of 3θ overlap with the values of 3θ found earlier, so in total there are $3 + 2 = 5$ values of 3θ in the interval $0^\circ \leq 3\theta \leq 540^\circ$, and hence 5 solutions to the original equation in the given interval. The correct answer is option E.

Question 10

Though integrating functions like this is not on the syllabus, the question assumes that we can calculate the value of this integral. There is also a hint from the options offered that it may have something to do with circles, as they all involve π.

Let us therefore consider the graph $y = \sqrt{4 - x^2}$. Squaring this gives $y^2 = 4 - x^2$ or $x^2 + y^2 = 4$. This is the equation of a circle with centre at the origin and radius 2. However, as the graph involves taking the square root of this, the resulting graph is just the upper semicircle (on and above the x-axis).



The integral is from $x = -2$ to $x = 2$, which is the whole diameter, so the integral is the area of the semicircle, that is, $\frac{1}{2} \times \pi \times 2^2 = 2\pi$.

We now need to calculate the area as determined by the trapezium rule with 4 strips. We can draw a table of values:

x	-2	-1	0	1	2
$\sqrt{4-x^2}$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

The trapezium rule then approximates the area as

$$\tfrac{1}{2} \times 1 \times \left(0 + 2 \times \sqrt{3} + 2 \times 2 + 2 \times \sqrt{3} + 0\right) = \tfrac{1}{2} \times \left(4 + 4\sqrt{3}\right) = 2 + 2\sqrt{3}.$$

The positive difference (as the trapezium rule underestimates the area) is therefore

$$2\pi - \left(2 + 2\sqrt{3}\right) = 2\left(\pi - 1 - \sqrt{3}\right)$$

which is option B.

Question 11

We first find the minimum points of the two graphs.

Approach 1: Direct approach

We have, for the first graph,

$$y = f(kx) = (kx)^2 - 6(kx) = k^2x^2 - 6kx.$$

We can find the minimum point using either calculus or completing the square; we will use calculus here. We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2k^2x - 6k$$

so $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ exactly when $x = \frac{3}{k}$. At this point, $y = k^2(\frac{3}{k})^2 - 6k(\frac{3}{k}) = -9$.

For the second graph, we have

$$y = f(x - c) = (x - c)^2 - 6(x - c) = x^2 - 2cx + c^2 - 6x + 6c = x^2 - (6 + 2c)x + c^2 + 6c.$$

We use completing the square to find the minimum point: we have

$$y = (x - (3 + c))^2 - (3 + c)^2 + c^2 + 6c = (x - (3 + c))^2 - 9$$

so the minimum point is at $x = 3 + c$, $y = -9$.

Approach 2: Using graph transformations

We first find the minimum point of $y = x^2 - 6x$. We can complete the square to get $y = (x - 3)^2 - 9$, so the minimum point is at $x = 3$, $y = -9$.

The graph $y = f(kx)$ is obtained by stretching the original graph by a factor of $1/k$ in the x-direction, so the minimum of this graph is at $x = 3/k$, $y = -9$.

The graph $y = f(x - c)$ is obtainedd by translating the original graph by c in the x-direction, so the minimum of this graph is at $x = 3 + c$, $y = -9$.

For the two minima to be at the same place, we require $3/k = 3 + c$, so taking the reciprocal gives

$$\frac{k}{3} = \frac{1}{3 + c}.$$

Multiplying by 3 then gives

$$k = \frac{3}{3 + c},$$

which is option B.

Question 12

Let us start by multiplying both sides by $4^{\sin^2 x}$ to remove the fraction:

$$2^{\tan^2 x} = 4^{\sin^2 x}.$$

We note that $4^a = (2^2)^a = 2^{2a}$, so we can rewrite this equation as

$$2^{\tan^2 x} = 2^{2\sin^2 x},$$

and therefore the equation can be simplified to

$$\tan^2 x = 2\sin^2 x.$$

Next, we know that $\tan x = \frac{\sin x}{\cos x}$, so we can rewrite this as

$$\frac{\sin^2 x}{\cos^2 x} = 2\sin^2 x.$$

Multiplying by $\cos^2 x$ to eliminate fractions once more gives

$$\sin^2 x = 2\sin^2 x \cos^2 x,$$

which we can rearrange and factorise to give

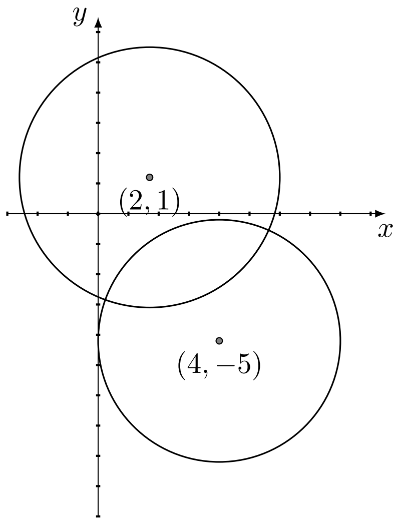
$$\sin^2 x(1 - 2\cos^2 x) = 0.$$

Therefore either $\sin x = 0$ or $\cos^2 x = \frac{1}{2}$. The first has solutions $x = 0$, π, 2π. The second gives $\cos x = \pm\frac{1}{\sqrt{2}}$, so has 4 solutions in the range $0 \le x \le 2\pi$, none of which are 0, π or 2π. (We could list them explicitly: they are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$.)

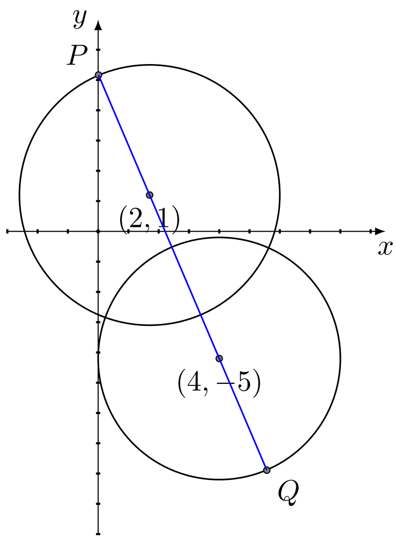
In total, there are 7 solutions, which is option F.

Question 13

It would help to sketch the circles. The first circle has centre $(2, 1)$ and radius 4; the second has centre $(4, -5)$ and radius 4. (This is not an accurate sketch, though we have noted that the second circle touches the y-axis.)



It is clear that the furthest apart that P and Q could be is if they are on the line that passes through the two circle centres as shown below:



The distance between P and Q is then the sum of the radius of the first circle, the distance between the centres and the radius of the second circle.

The distance between the centres is $\sqrt{(4-2)^2 + (-5-1)^2} = \sqrt{40} = 2\sqrt{10}$ and each radius is 4, so the maximum distance is $8 + 2\sqrt{10}$, which is option F.

Question 14

The function will have three distinct real roots if and only if the graph $y = f(x)$ has two stationary points, one above the x-axis and one below. We first find the stationary points. We have

$$f'(x) = 2x^2 + 4mx = 2x(x + 2m)$$

so $f'(x) = 0$ when $x = 0$ or $x = -2m$.

When $x = 0$, $f(x) = n$, and when $x = -2m$,

$$f(x) = \frac{2}{3}(-8m^3) + 2m(4m^2) + n = n + \frac{8}{3}m^3.$$

Since one of the stationary points has $f(x) = n$, we cannot have $n = 0$.

Since $m > 0$, it follows that if $n > 0$, then $f(-2m) > 0$ and both turning points would be above the x-axis. We must therefore have $n < 0$ and $n + \frac{8}{3}m^3 > 0$. The second inequality rearranges to $-\frac{8}{3}m^3 < n$, so we require $-\frac{8}{3}m^3 < n < 0$, which is option A.

Question 15

The minimum and maximum values occur when $\cos x = \pm 1$ (though which corresponds to the maximum and which to the minimum depends on whether $a < 1$ or $a > 1$). Therefore the minimum and maximum values are a and $1/a$ in some order.

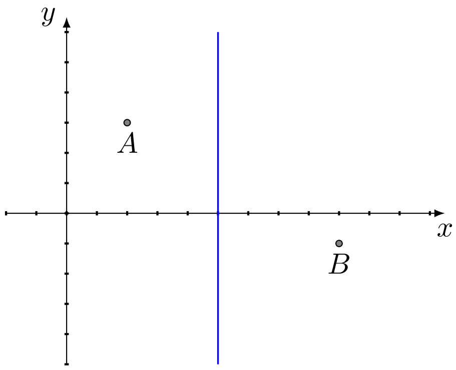
Assuming $a > 1$, so that $a > 1/a$, we require $a - \frac{1}{a} = 3$. Multiplying by a and rearranging gives $a^2 - 3a - 1 = 0$, which has roots $a = \frac{3 \pm \sqrt{13}}{2}$. The minus sign would give $a < 0$, so the root we want is $a = \frac{3+\sqrt{13}}{2}$, and this clearly satisfies $a > 1$.

Now assuming $a < 1$, so that $1/a > a$, we require $\frac{1}{a} - a = 3$. Multiplying by a and rearranging gives $a^2 + 3a - 1 = 0$, which has roots $a = \frac{-3 \pm \sqrt{13}}{2}$. The minus sign would give $a < 0$, so the root we want is $a = \frac{-3+\sqrt{13}}{2}$ (and as $3 < \sqrt{13} < 4$, this lies between 0 and $\frac{1}{2}$, hence $a < 1$).

The sum of these two values is $\frac{3+\sqrt{13}}{2} + \frac{-3+\sqrt{13}}{2} = \sqrt{13}$, which is option F.

Question 16

A sketch is useful here; we also label the points as $A(2,3)$, $B(9,-1)$ and $C(5,k)$. The point C lies on the blue line in this sketch.



We could have C at about $(5,5)$ and have a right angle at A. We could have C a little lower and have a right angle at C. We could have C at about $(5,-5)$ and have a right angle at C, or we could have C a little lower and have a right angle at B. We work through these four possibilities.

- Right angle at A. The gradient of the line AB is $\frac{-1-3}{9-2} = -\frac{4}{7}$, so the gradient of the line AC is $\frac{7}{4}$.

  The equation of the line AC is thus $y-3 = \frac{7}{4}(x-2)$. This intersects the line $x=5$ when $y-3 = \frac{7}{4}(5-2)$, giving $y = 3 + \frac{21}{4} = \frac{33}{4}$. So $k = \frac{33}{4}$ gives a right-angled triangle.

- Right angle at B. As in the first case, the gradient of the line BC is $\frac{7}{4}$.

  The equation of the line BC is thus $y+1 = \frac{7}{4}(x-9)$. This intersects the line $x=5$ when $y+1 = \frac{7}{4}(5-9)$, giving $y = -1-7 = -8$. So $k=-8$ gives a right-angled triangle.

- Right angle at C. We could determine the possible values of k either by observing that the lines AC and BC must be perpendicular, or by using Pythagoras's Theorem to solve $AC^2 + BC^2 = AC^2$. Let us use the perpendicular approach here. We have

$$\text{gradient of } AC = \frac{k-3}{5-2} = \frac{k-3}{3}$$
$$\text{gradient of } BC = \frac{k+1}{5-9} = -\frac{k+1}{4}$$

  These must multiply to -1, so

$$\frac{k-3}{3} \cdot \frac{k+1}{4} = 1$$

  which simplifies to $(k-3)(k+1) = 12$. We can solve this as a quadratic as usual: it rearranges to $k^2 - 2k - 15 = 0$ giving $k=5$ and $k=-3$.

Adding these possible values of k gives a total of $\frac{33}{4} - 8 + 5 - 3 = \frac{9}{4}$, which is option E.

Question 17

It seems from the given sketch that each of the circles passes through the origin, and that all the circles are tangent to the line $y = -x$ at the origin. This means that the shaded area between C_n and C_{n+1} is just the area of C_{n+1} minus the area of C_n, as C_n lies entirely within C_{n+1} except for the single point of tangency at the origin.

We can prove this. Firstly, the circle C_n passes through the origin as $(x, y) = (0, 0)$ satisfies the equation of the circle. Next, we can solve the equation of the circle C_n simultaneously with $y = -x$. Substituting $y = -x$ into the equation of the circle gives

$$x^2 + x^2 = 2n(x + (-x)) = 0$$

so $2x^2 = 0$, which has a repeated root $x = 0$. Therefore this line is tangent to the circle at the origin.

Next, we need to find the area of C_n. We rearrange and complete the square to do this: we have

$$\begin{aligned} x^2 - 2nx + y^2 - 2ny &= 0 \\ \text{so} \quad (x-n)^2 - n^2 + (y-n)^2 - n^2 &= 0 \\ \text{so} \quad (x-n)^2 + (y-n)^2 &= 2n^2 \end{aligned}$$

This shows that the centre of C_n is at (n, n) and it has squared radius $2n^2$. Its area is therefore $2\pi n^2$.

We deduce that the shaded area between C_{n+1} and C_n is

$$2\pi(n+1)^2 - 2\pi n^2 = 2\pi(2n+1).$$

Summing this for $n = 1, 3, 5, \ldots, 99$ (as we want the difference between circles C_1 and C_2, and between C_3 and C_4, and so on), the total shaded area is

$$2\pi(3 + 7 + 11 + \cdots + 199)$$

The sum here is an arithmetic series; we have first term $a = 3$, last term $l = 199$, and $n = 50$ terms (as there are 50 pairs of circles), so the total area is

$$2\pi \times \tfrac{1}{2}n(a+l) = \pi \times 50 \times 202 = 10100\pi$$

which is option E.

Question 18

The sum is an infinite geometric series with first term $a = 4$ and common ratio $r = \frac{2k}{7}$. For this series to have a finite sum, we require $|r| < 1$, so $-7 < 2k < 7$, that is $-\frac{7}{2} < k < \frac{7}{2}$. Thus we can only consider $k = -3, -2, \ldots, 2, 3$ out of the 11 possible values of k. (Note that there are 11 values: $k = 0$ is a possibility!)

In these cases, the sum is given by

$$S = \frac{a}{1-r} = \frac{4}{1-2k/7} = \frac{28}{7-2k}.$$

We need to solve the inequality $S > 3$, so

$$\frac{28}{7-2k} > 3.$$

Since $7 - 2k > 0$ for all $k < \frac{7}{2}$, which are the only values of k of interest to us, we can multiply by $7 - 2k$ to get

$$28 > 3(7 - 2k).$$

Expanding and rearranging gives $6k > -7$, so $k > -\frac{7}{6}$. Therefore the values of k that satisfy the conditions are -1, 0, 1, 2, 3, a total of 5 values out of the 11 possible values. Therefore the required probability is $\frac{5}{11}$, which is option E.

Question 19

One thing we might notice is that $|-6x| = |6x|$, so the minus sign in the differential equation is not actually needed. We therefore drop it.

A more important thing to realise is that we should not try to solve the whole equation in one go; we will need to treat $x \geq 0$ and $x \leq 0$ separately.

With that in mind, let us solve the equation.

When $x \geq 0$, the equation becomes $\frac{dy}{dx} = 6x$, so $y = 3x^2 + c$.

When $x \leq 0$, the equation becomes $\frac{dy}{dx} = -6x$, so $y = -3x^2 + \hat{c}$. (We use a different constant, $\hat{c}$, as we do not yet know whether it has the same value as the constant for the $x \geq 0$ region.)

Since the value of y has to be the same in both expressions when $x = 0$, we must have $\hat{c} = c$, so we have

$$y = \begin{cases} 3x^2 + c & \text{when } x \geq 0 \\ -3x^2 + c & \text{when } x < 0 \end{cases}$$

giving

$$f(x) = \begin{cases} 3x^2 & \text{when } x \geq 0 \\ -3x^2 & \text{when } x < 0 \end{cases}$$

The options offered require us to collapse this into a single expression. We can take out a common factor of $3x$ from both terms, to give

$$f(x) = \begin{cases} 3x(x) & \text{when } x \geq 0 \\ 3x(-x) & \text{when } x < 0 \end{cases}$$

and this is $3x$ times $|x|$, since

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

So we can write $f(x) = 3x\,|x|$, which is option D.

Question 20

Let us first consider the function $y = g(x)$, where $g(x) = x^2 - x$; we will then substitute $f(x)$ into this expression once we understand the behaviour of $y = g(x)$.

$g(x)$ is a quadratic and we can complete the square: $g(x) = (x - \frac{1}{2})^2 - \frac{1}{4}$. So $y = g(x)$ has a minimum value of $y = -\frac{1}{4}$ at $x = \frac{1}{2}$.

When we now substitute the values of $f(x)$ into this completed-square form for $g(x)$, obtaining $\big(f(x)\big)^2 - f(x)$. We can see from the graph that there is some value of x with $f(x) = \frac{1}{2}$, so this expression has minimum value of $-\frac{1}{4}$.

For the maximum value, we see that the value of $g(x)$ increases symmetrically either side of $x = \frac{1}{2}$. So the maximum value of $\big(f(x)\big)^2 - f(x)$ will either occur when $f(x) = 2$ or when $f(x) = -2$. Since -2 is further from $\frac{1}{2}$ than 2 is, $f(x) = -2$ will give the maximum value, which is $(-2)^2 - (-2) = 6$. (And if we hadn't made that observation, we could have calculated $2^2 - 2 = 2$ and seen that it is smaller than 6.)

Therefore the difference between the maximum and minimum values is $6 - (-\frac{1}{4}) = \frac{25}{4}$, which is option F.